

On the Topological Pressure of Random Bundle Transformations in Sub-additive Case

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Abstract. In this paper, we define the topological pressure for sub-additive potential via separated sets in random dynamical systems and give a proof of the relativized variational principle for the topological pressure.

Key words and phrases variational principle, topological pressure, entropy.

1 Introduction.

The setup consists of a probability space $(\Omega, \mathcal{W}, \mathbb{P})$, together with a \mathbb{P} -preserving transformation ϑ , of a compact metric space X together with the distance function d and the Borel σ -algebra \mathcal{B}_X , and of a measurable set $\mathcal{E} \subset \Omega \times X$ and such that all the fibers (sometimes called ω -sections) $\mathcal{E}_\omega = \{x \in X \mid (\omega, x) \in \mathcal{E}\}$ are compact. We assume \mathcal{W} is complete, countably generated, and separates points, and so $(\Omega, \mathcal{W}, \mathbb{P})$ is a Lebesgue space. A continuous bundle RDS over $(\Omega, \mathcal{W}, \mathbb{P}, \vartheta)$ is generated by mappings $T_\omega : \mathcal{E}_\omega \rightarrow \mathcal{E}_{\vartheta\omega}$ with iterates $T_\omega^n = T_{\vartheta^{n-1}\omega} \cdots T_{\vartheta\omega} T_\omega$, $n \geq 1$, so that the map $(\omega, x) \mapsto T_\omega x$ is measurable and the map $x \mapsto T_\omega x$ is continuous for \mathbb{P} -almost all ω , here and in what follows we think of \mathcal{E}_ω being equipped with the trace topology, i.e. an open set $A \subset \mathcal{E}_\omega$ is of the form $A = B \cap \mathcal{E}_\omega$ with some open set $B \subset X$. The map

$$\Theta : \mathcal{E} \rightarrow \mathcal{E}, \Theta(\omega, x) = (\vartheta\omega, T_\omega x)$$

is called the skew product transformation.

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Let $L_{\mathcal{E}}^1(\Omega, C(X))$ denote the collection of all integrable random continuous functions on fibers, i.e. a measurable $f : \mathcal{E} \rightarrow \mathbf{R}$ is a member of $L_{\mathcal{E}}^1(\Omega, C(X))$ if $f(\omega, \cdot) : \mathcal{E}_\omega \rightarrow \mathbf{R}$ is continuous and $\|f\| := \int_{\Omega} |f(\omega)|_\infty d\mathbb{P}(\omega) < \infty$, where $|f(\omega)|_\infty = \sup_{x \in \mathcal{E}_\omega} |f(\omega, x)|$. If we identify f and g provided $\|f - g\| = 0$, then $L_{\mathcal{E}}^1(\Omega, C(X))$ becomes a Banach space with the norm $\|\cdot\|$.

The family $\mathcal{F} = \{f_n\}_{n=1}^\infty$ of integrable random continuous functions on \mathcal{E} is called sub-additive if for \mathbb{P} -almost all ω ,

$$f_{n+m}(\omega, x) \leq f_n(\omega, x) + f_m(\Theta^n(\omega, x)) \text{ for all } x \in \mathcal{E}_\omega.$$

In the special case in which the ϑ -invariant measure \mathbb{P} is a Dirac- δ measure supported on a single fixed point $\{p\}$, it reduces to the case in which $T : X \rightarrow X$ is a standard deterministic dynamical system.

In deterministic dynamical systems $T : X \rightarrow X$, the topological pressure for additive potential was first introduced by Ruelle [15] for expansive maps acting on compact metric spaces. In the same paper he formulated a variational principle for the topological pressure. Later Walters [16] generalized these results to general continuous maps on compact metric spaces. The theory about the topological pressure, variational principle and equilibrium states plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems. The fact that the topological pressure is a characteristic of dimension type was first noticed by Bowen [5]. Since then, it has become the main tool in studying dimension of invariant sets and measure for dynamical systems and the dimension of cantor-like sets in dimension theory.

In [6], authors generalize Ruelle and Walters's result to sub-multiplicative potentials in general compact dynamical systems. They define the sub-multiplicative topological pressure and give a variational principle for the sub-multiplicative topological pressure. Then in [7], author uses the variational principle for the sub-multiplicative topological pressure to give an upper bound estimate of Hausdorff dimension for nonconformal repeller, which generalizes the results by Falconer in [8], Barreira in [1, 2], and Zhang in [18].

We point out that Falconer had some earlier contributions in the study of thermodynamic formalism for sub-additive potentials. In [9], Falconer considered the thermodynamic formalism for sub-additive potentials on mixing repellers. He proved the variational principle about the topological pressure under some Lipschitz conditions and bounded distortion assumptions on the sub-additive potentials. More precisely, he assumed that there exist constants $M, a, b > 0$ such that

$$\frac{1}{n} |\log f_n(x)| \leq M, \quad \frac{1}{n} |\log f_n(x) - \log f_n(y)| \leq a|x - y|, \quad \forall x, y \in X, n \in \mathcal{N}$$

and $|\log f_n(x) - \log f_n(y)| \leq b$ whenever x, y belong to the same n -cylinder of the mixing repeller X .

In deterministic case, the thermodynamic formalism based on the statistical mechanics notions of pressure and equilibrium states plays an important role in the study of chaotic properties of random transformations. The first version of the relativized variational principle appeared in [13] and later it was extended in [3] to random transformations for special potential function. In [11], Kifer extended the variational principle of topological pressure for general integrable random continuous function.

The aim of this paper is to introduce topological pressure of random bundle transformations for sub-additive potentials, and show a relativized variational principle. We can see it as an extension of results in [6], [11]. The paper is organized in the following manner: in section 2 we introduce the definitions. In section 3 we will provide some useful lemmas. In section 4 we will state and prove the main theorem: the relativized variational principle. In section 5 we will apply topological pressure of random bundle transformations for sub-additive potentials to obtain the Hausdorff dimension of asymptotically conformal repeller.

2 Topological pressure and entropy of bundle RDS

In this section, we give the definitions of entropy and the topological pressure for sub-additive potential.

Denote by $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$ the space of probability measures on $\Omega \times X$ having the marginal \mathbb{P} on Ω and set $\mathcal{P}_{\mathbb{P}}(\mathcal{E}) = \{\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X) : \mu(\mathcal{E}) = 1\}$. Any $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ on \mathcal{E} disintegrates $d\mu(\omega, x) = d\mu_{\omega}(x)d\mathbb{P}(\omega)$, where μ_{ω} are regular conditional probabilities with respect to the σ -algebra $\mathcal{W}_{\mathcal{E}}$ formed by all sets $(A \times X) \cap \mathcal{E}$ with $A \in \mathcal{W}$. This means that μ_{ω} is a probability measure on \mathcal{E}_{ω} for $\mathbb{P} - a.a.\omega$ and for any measurable set $R \subset \mathcal{E}$, \mathbb{P} -a.s. $\mu_{\omega}(R_{\omega}) = \mu(R|\mathcal{W}_{\mathcal{E}})$, where $R_{\omega} = \{x : (\omega, x) \in R\}$, and so $\mu(R) = \int \mu_{\omega}(R_{\omega})d\mathbb{P}(\omega)$. Now let $\mathcal{R} = \{R_i\}$ be a finite or countable partition of \mathcal{E} into measurable sets. Then $\mathcal{R}(\omega) = \{R_i(\omega)\}$, $R_i(\omega) = \{x \in \mathcal{E}_{\omega} : (\omega, x) \in R_i\}$ is a partition of \mathcal{E}_{ω} . The conditional entropy of \mathcal{R} given the σ -algebra $\mathcal{W}_{\mathcal{E}}$ is defined by

$$H_{\mu}(\mathcal{R} \mid \mathcal{W}_{\mathcal{E}}) = - \int \sum_i \mu(R_i \mid \mathcal{W}_{\mathcal{E}}) \log \mu(R_i \mid \mathcal{W}_{\mathcal{E}}) d\mathbb{P} \quad (2.1)$$

$$= \int H_{\mu_{\omega}}(\mathcal{R}(\omega)) d\mathbb{P} \quad (2.2)$$

where $H_{\mu_{\omega}}(\mathcal{A})$ denotes the usual entropy of a partition \mathcal{A} . Let $\mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$ denote the set of Θ -invariant measures $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$. The entropy $h_{\mu}^{(r)}(T)$ of the RDS T with respect to μ is defined by the formula

$$h_{\mu}^{(r)}(T) = \sup_{\mathcal{Q}} h_{\mu}^{(r)}(T, \mathcal{Q}) \quad (2.3)$$

where $h_\mu^{(r)}(T, \mathcal{Q}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} (\Theta^i)^{-1} \mathcal{Q} \mid \mathcal{W}_\mathcal{E} \right)$ and the supremum is taken over all finite or countable measurable partitions $\mathcal{Q} = \{Q_i\}$ of \mathcal{E} with $H_\mu(\mathcal{Q} \mid \mathcal{W}_\mathcal{E}) < \infty$.

Observe that if $\mathcal{Q} = \{Q_i\}$ is a partition of \mathcal{E} , then $\mathcal{R} = \bigvee_{i=0}^{n-1} (\Theta^i)^{-1} \mathcal{Q}$ is a partition of \mathcal{E} consisting of sets $\{R_j\}$ such that the corresponding partition $\mathcal{R}(\omega) = \{R_j(\omega)\}$, $R_j(\omega) = \{x : (\omega, x) \in R_j\}$ of \mathcal{E}_ω has the form $\mathcal{R}(\omega) = \bigvee_{i=0}^{n-1} (T_\omega^i)^{-1} \mathcal{Q}(\vartheta^i \omega)$, where $\mathcal{Q}(\omega) = \{Q_i(\omega)\}$, $Q_i(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in Q_i\}$ is a partition of \mathcal{E}_ω . So

$$h_\mu^{(r)}(T, \mathcal{Q}) = \lim_{n \rightarrow \infty} \frac{1}{n} \int H_{\mu_\omega} \left(\bigvee_{i=0}^{n-1} (T_\omega^i)^{-1} \mathcal{Q}(\vartheta^i \omega) \right) d\mathbb{P}(\omega) \quad (2.4)$$

In [3] and [10], the authors say that the resulting entropy remains the same if we take the supremum in (2.3) only over partitions \mathcal{Q} of \mathcal{E} into sets Q_i of the form $Q_i = (\Omega \times A_i) \cap \mathcal{E}$, where $\mathcal{A} = \{A_i\}$ is a partition of X into measurable sets, so that $Q_i(\omega) = A_i \cap \mathcal{E}_\omega$. If ϑ is invertible, then $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E})$ is Θ -invariant if and only if the disintegrations μ_ω of μ satisfy $T_\omega \mu_\omega = \mu_{\vartheta \omega} \mathbb{P} - a.s.$ In this case, if, in addition, \mathbb{P} is ergodic, then the formula (2.4) remains true \mathbb{P} -a.s. without integrating against \mathbb{P} .

For each $n \in \mathbf{N}$ and a positive random variable $\epsilon = \epsilon(\omega)$, we define a family of metrics $d_{\epsilon,n}^\omega$ on \mathcal{E}_ω by the formula

$$d_{\epsilon,n}^\omega(x, y) = \max_{0 \leq k < n} (d(T_\omega^k y, T_\omega^k x) \times (\epsilon(\vartheta^k \omega))^{-1}), \quad x, y \in \mathcal{E}_\omega$$

where T_ω^0 is the identity map. In [11], the author proves that $d_{\epsilon,n}^\omega(x, y)$ depends measurably on $(\omega, x, y) \in \mathcal{E}^{(2)} := \{(\omega, x, y) : x, y \in \mathcal{E}_\omega\}$. Denote by $B_\omega(n, x, \epsilon)$ the closed ball in \mathcal{E}_ω centered at x of radius 1 with respect to the metric $d_{\epsilon,n}^\omega$. For $d_{\epsilon,1}^\omega$ and $B_\omega(1, x, \epsilon)$, we will write simply d_ϵ^ω and $B_\omega(x, \epsilon)$ respectively. We say that $x, y \in \mathcal{E}_\omega$ are (ω, ϵ, n) -close if $d_{\epsilon,n}^\omega(x, y) \leq 1$.

Definition 2.1. A set $F \subset \mathcal{E}_\omega$ is said to be (ω, ϵ, n) -separated for T , if $x, y \in F, x \neq y$ implies $d_{\epsilon,n}^\omega(x, y) > 1$.

It is easy to see that if F is maximal (ω, ϵ, n) -separated, i.e. for every $x \in \mathcal{E}_\omega$ with $x \notin F$ the set $F \cup \{x\}$ is not (ω, ϵ, n) -separated anymore, then $\mathcal{E}_\omega = \bigcup_{x \in F} B_\omega(n, x, \epsilon)$. Due to the compactness of \mathcal{E}_ω , there exists a maximal (ω, ϵ, n) -separated set F with finite elements.

Let $\mathcal{F} = \{f_n\}$ be a sub-additive function sequence with $f_n \in L_\mathcal{E}^1(\Omega, C(X))$ for each n . As usual for any $n \in \mathbf{N}$ and a positive random variable ϵ , we define

$$\pi_T(\mathcal{F})(\omega, \epsilon, n) = \sup \left\{ \sum_{x \in F} e^{f_n(\omega, x)} \mid F \text{ is an } (\omega, \epsilon, n) \text{-separated subset of } \mathcal{E}_\omega \right\}$$

and

$$\begin{aligned}\pi_T(\mathcal{F})(\epsilon) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \int \log \pi_T(\mathcal{F})(\omega, \epsilon, n) d\mathbb{P}(\omega), \\ \pi_T(\mathcal{F}) &= \lim_{\epsilon \downarrow 0} \pi_T(\mathcal{F})(\epsilon).\end{aligned}$$

By lemma 3.1 in section 3, we know that the definition of $\pi_T(\mathcal{F})(\epsilon)$ is reasonable. The last limit exists since $\pi_T(\mathcal{F})(\epsilon)$ is monotone in ϵ . In fact, $\lim_{\epsilon \rightarrow 0}$ as above equals to $\sup_{\epsilon > 0}$.

Remark 1. *In [11], the author defined additive topological pressure for a random positive variable ϵ , but the limit should be taken over some directed sets. We can find detailed description of difference between random and nonrandom case of ϵ in [4].*

3 Some Lemmas

In this section, we will give some lemmas which will be used in our proof of the main theorem in the next section.

Let \mathcal{F}, T and $\pi_T(\mathcal{F})$ be defined as in section 2, and (X, d) be a compact metric space. Notice that if $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$, then we let $\mathcal{F}_*(\mu)$ denote the following limit $\mathcal{F}_*(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathcal{E}} f_n d\mu$. The existence of the limit follows from a sub-additive argument. We begin with the following lemmas, and we point out that the proof of the first two lemmas can be easily obtained by following the proof in [11]. We cite here just for complete.

Lemma 3.1. *For any $n \in \mathbb{N}$ and a positive random variable $\epsilon = \epsilon(\omega)$ the function $\pi_T(\mathcal{F})(\omega, \epsilon, n)$ is measurable in ω , and for each $\delta > 0$ there exists a family of maximal (ω, ϵ, n) separated sets $G_{\omega} \subset \mathcal{E}_{\omega}$ satisfying*

$$\sum_{x \in G_{\omega}} e^{f_n(\omega, x)} \geq (1 - \delta) \pi_T(\mathcal{F})(\omega, \epsilon, n)$$

and depending measurably on ω in the sense that $G = \{(\omega, x) : x \in G_{\omega}\} \in \mathcal{W} \times \mathcal{B}_X$, which also means that the mapping $\omega \mapsto G_{\omega}$ is measurable with respect to the Borel σ -algebra induced by the Hausdorff topology on the space $\mathcal{K}(X)$ of compact subsets of X . In particular, the supremum in the definition of $\pi_T(\mathcal{F})(\omega, \epsilon, n)$ can be taken only over measurable in ω families of (ω, ϵ, n) -separated sets.

Lemma 3.2. *For $\mu, \mu_n \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, $n = 1, 2, \dots$, write $\mu_n \Rightarrow \mu$ if $\int f d\mu_n \rightarrow \int f d\mu$ as $n \rightarrow \infty$ for any $f \in L_{\mathcal{E}}^1(\Omega, C(X))$ that introduces a weak* topology in $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$. Then*

(i) the space $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ is compact in this weak topology;*

(ii) for any sequence $v_k \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, $k = 0, 1, 2, \dots$, the set of limit points in the above weak* topology of the sequence

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \Theta^k v_n \quad \text{as } n \rightarrow \infty$$

is not empty and is contained in $\mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$;

(iii) let $\mu, \mu_n \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, $n = 1, 2, \dots$, and $\mu_n \Rightarrow \mu$ as $n \rightarrow \infty$; let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a finite partition of X satisfying $\int \mu_{\omega}(\partial \mathcal{P}_{\omega}) d\mathbb{P}(\omega) = 0$, where $\partial \mathcal{P}_{\omega} = \bigcup_{i=1}^k \partial(P_i \cap \mathcal{E}_{\omega})$ is the boundary of $\mathcal{P}_{\omega} = \{P_1 \cap \mathcal{E}_{\omega}, \dots, P_k \cap \mathcal{E}_{\omega}\}$; denote by \mathcal{R} the partition of $\Omega \times X$ into sets $\Omega \times P_i$; then

$$\limsup_{n \rightarrow \infty} H_{\mu_n}(\mathcal{R} \mid \mathcal{W}_{\mathcal{E}}) \leq H_{\mu}(\mathcal{R} \mid \mathcal{W}_{\mathcal{E}})$$

Lemma 3.3. For any $k \in \mathbf{N}$, we have

$$\pi_{T^k}(\mathcal{F}^{(k)}) \leq k\pi_T(\mathcal{F})$$

where $(T^k)_{\omega} := T_{\vartheta^{k-1}\omega} \circ \dots \circ T_{\vartheta\omega} \circ T_{\omega}$ and $\mathcal{F}^{(k)} := \{f_{kn}\}_{n=1}^{\infty}$.

Proof. Fix $k \in \mathbf{N}$. Note that if F is an (ω, ϵ, n) separated set for T^k of \mathcal{E}_{ω} , then F is an (ω, ϵ, kn) separated set for T of \mathcal{E}_{ω} . It follows that

$$\begin{aligned} \pi_T(\mathcal{F})(\omega, \epsilon, kn) &= \sup \left\{ \sum_{x \in F} e^{f_{kn}(\omega, x)} : F \text{ is an } (\omega, \epsilon, kn) \text{ separated subset of } \mathcal{E}_{\omega} \text{ for } T \right\} \\ &\geq \sup \left\{ \sum_{x \in F} e^{f_{kn}(\omega, x)} : F \text{ is an } (\omega, \epsilon, n) \text{ separated subset of } \mathcal{E}_{\omega} \text{ for } T^k \right\} \\ &= \pi_{T^k}(\mathcal{F}^k)(\omega, \epsilon, n). \end{aligned}$$

It implies that $\pi_{T^k}(\mathcal{F}^k) \leq k\pi_T(\mathcal{F})$. □

Lemma 3.4. For any positive integer k and $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, we have

$$\int_{\mathcal{E}} k f_n(\omega, x) d\mu \leq 4k^2 C + \int_{\mathcal{E}} \sum_{i=0}^{n-1} f_k(\Theta^i(\omega, x)) d\mu$$

where $C = \|f_1\|$.

Proof. For a fixed k , it has $n = ks + l$, $0 \leq l < k$. For $j = 0, 1, \dots, k-1$, the subadditivity of $f_n(\omega, x)$ implies that

$$f_n(\omega, x) \leq f_j(\omega, x) + f_k(\Theta^j(\omega, x)) + \dots + f_k(\Theta^{k(s-2)}\Theta^j(\omega, x)) + f_{k+l-j}(\Theta^{k(s-1)}\Theta^j(\omega, x)).$$

Hence

$$\begin{aligned}
\int_{\mathcal{E}} f_n(\omega, x) d\mu &\leq \int_{\mathcal{E}} f_j(\omega, x) d\mu + \int_{\mathcal{E}} \sum_{i=0}^{s-2} f_k(\Theta^{ki} \Theta^j(\omega, x)) d\mu + \int_{\mathcal{E}} f_{k+l-j}(\Theta^{k(s-1)} \Theta^j(\omega, x)) d\mu \\
&\leq \|f_j\| + \int_{\mathcal{E}} \sum_{i=0}^{s-2} f_k(\Theta^{ki} \Theta^j(\omega, x)) d\mu + \|f_{k+l-j}\| \\
&\leq 2k\|f_1\| + \int_{\mathcal{E}} \sum_{i=0}^{s-2} f_k(\Theta^{ki} \Theta^j(\omega, x)) d\mu.
\end{aligned}$$

Summing j from 0 to $k-1$, we get

$$\begin{aligned}
\int_{\mathcal{E}} kf_n(\omega, x) d\mu &\leq 2k^2C + \int_{\mathcal{E}} \sum_{i=0}^{k(s-1)-1} f_k(\Theta^i(\omega, x)) d\mu \\
&= 2k^2C + \int_{\mathcal{E}} \sum_{i=0}^{n-1} f_k(\Theta^i(\omega, x)) d\mu - \int_{\mathcal{E}} \sum_{i=k(s-1)}^{n-1} f_k(\Theta^i(\omega, x)) d\mu \\
&\leq 4k^2C + \int_{\mathcal{E}} \sum_{i=0}^{n-1} f_k(\Theta^i(\omega, x)) d\mu.
\end{aligned}$$

This finishes the proof of the lemma. \square

Lemma 3.5. *Let $m^{(n)}$ be a sequence in $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$. The new sequence $\{\mu^{(n)}\}_{n=1}^{\infty}$ is defined as $\mu^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} \Theta^i m^{(n)}$. Assume $\mu^{(n_i)}$ converges to μ in $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ for some subsequence $\{n_i\}$. Then $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$, and moreover*

$$\limsup_{i \rightarrow \infty} \frac{1}{n_i} \int_{\mathcal{E}} f_{n_i}(\omega, x) dm^{(n_i)}(\omega, x) \leq \mathcal{F}_*(\mu)$$

where $\mathcal{F}_*(\mu) = \inf_n \left\{ \frac{1}{n} \int_{\mathcal{E}} f_n(\omega, x) d\mu \right\}$.

Proof. The first statement $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$ is contained in lemma 3.2. To show the desired inequality, we fix $k \in \mathbf{N}$. By lemma 3.4, we have

$$\begin{aligned}
\frac{1}{n} \int_{\mathcal{E}} f_n(\omega, x) dm^{(n)} &= \frac{1}{kn} \int_{\mathcal{E}} kf_n(\omega, x) dm^{(n)} \\
&\leq \frac{1}{kn} (4k^2C + \int_{\mathcal{E}} \sum_{j=0}^{n-1} f_k(\Theta^j(\omega, x)) dm^{(n)}) \\
&= \frac{4kC}{n} + \int_{\mathcal{E}} \frac{1}{k} f_k(\omega, x) d\mu^{(n)}.
\end{aligned}$$

In particular, we have

$$\frac{1}{n_i} \int_{\mathcal{E}} f_{n_i}(\omega, x) dm^{(n_i)} \leq \int_{\mathcal{E}} \frac{1}{k} f_k(\omega, x) d\mu^{(n_i)} + \frac{4kC}{n_i}.$$

Since $\lim_{i \rightarrow \infty} \mu^{(n_i)} = \mu$, we have

$$\limsup_{i \rightarrow \infty} \frac{1}{n_i} \int_{\mathcal{E}} f_{n_i}(\omega, x) dm^{(n_i)} \leq \int_{\mathcal{E}} \frac{1}{k} f_k(\omega, x) d\mu.$$

Letting k approach infinity and applying the sub-additive ergodic theorem, we have the desired result. \square

4 The statement of main theorem and its proof

For random dynamical systems, the topological pressure for sub-additive potential also has variational principle which can be considered as a generalization of variational principle of topological pressure for sub-additive potential in deterministic dynamical systems in [6]. Next we give a statement of main theorem and its proof.

Theorem 4.1. *Let Θ be a continuous bundle random dynamical systems on \mathcal{E} , and \mathcal{F} a sequence of sub-additive random continuous functions in $L_{\mathcal{E}}^1(\Omega, C(X))$. Then*

$$\pi_T(\mathcal{F}) = \begin{cases} -\infty, & \text{if } \mathcal{F}_*(\mu) = -\infty \text{ for all } \mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T) \\ \sup\{h_{\mu}^{(r)}(T) + \mathcal{F}_*(\mu) : \mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)\}, & \text{otherwise.} \end{cases}$$

Proof. For clarity, we divide the proof into three small steps:

Step 1: $\pi_T(\mathcal{F}) \geq h_{\mu}^{(r)}(T) + \mathcal{F}_*(\mu)$, $\forall \mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$ with $\mathcal{F}_*(\mu) \neq -\infty$.

Let $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$ satisfying $\mathcal{F}_*(\mu) \neq -\infty$ and $\mathcal{A} = \{A_1, \dots, A_k\}$ be a finite partition of X . Let $\alpha > 0$ be given. Choose $\epsilon > 0$ so that $\epsilon k \log k < \alpha$. Denote by $\mathcal{A}(\omega) = \{A_1(\omega), \dots, A_k(\omega)\}$, $A_i(\omega) = A_i \cap \mathcal{E}_{\omega}$, $i = 1, \dots, k$, the corresponding partition of \mathcal{E}_{ω} . By the regularity of μ , we can find compact sets $B_i \subset A_i$, $1 \leq i \leq k$, such that

$$\mu(A_i \setminus B_i) = \int \mu_{\omega}(A_i(\omega) \setminus B_i(\omega)) d\mathbb{P}(\omega) < \epsilon,$$

where $B_i(\omega) = B_i \cap \mathcal{E}_{\omega}$. Then let $B_0(\omega) = \mathcal{E}_{\omega} \setminus \bigcup_{i=1}^k B_i(\omega)$. It follows that

$$\int \mu_{\omega}(B_0(\omega)) d\mathbb{P}(\omega) < k\epsilon.$$

Therefore (see [10] p.79) the partition $\mathcal{B}(\omega) = \{B_0(\omega), \dots, B_k(\omega)\}$ satisfies the inequality

$$H_{\mu_{\omega}}(\mathcal{A}(\omega) \mid \mathcal{B}(\omega)) \leq \mu_{\omega}(B_0(\omega)) \log k.$$

Hence

$$\int H_{\mu_{\omega}}(\mathcal{A}(\omega) \mid \mathcal{B}(\omega)) d\mathbb{P} \leq \int \mu_{\omega}(B_0(\omega)) \log k d\mathbb{P} \leq k\epsilon \log k < \alpha.$$

Take any ω such that \mathcal{E}_{ω} makes sense, and set $b = \min_{1 \leq i \neq j \leq k} d(B_i, B_j) > 0$. Pick $\delta > 0$ so that $\delta < b/2$. Let $n \in \mathbf{N}$. For each $C \in \mathcal{B}_n(\omega) := \bigvee_{j=0}^{n-1} (T_w^j)^{-1} \mathcal{B}(\vartheta^j(\omega))$, choose some

$x(C) \in \text{Closure}(C)$ such that $f_n(\omega, x(C)) = \sup\{f_n(\omega, x) : x \in C\}$, and we claim that for each $C \in \mathcal{B}_n(\omega)$, there are at most 2^n many different \tilde{C} 's in $\mathcal{B}_n(\omega)$ such that

$$d_{\delta, n}^\omega(x(C), x(\tilde{C})) := \max_{0 \leq j \leq n-1} d(T_\omega^j x(C), T_\omega^j x(\tilde{C})) \delta^{-1} \leq 1.$$

To see this claim, for each $C \in \mathcal{B}_n(\omega)$ we pick up the unique index $(i_0(C), i_1(C), \dots, i_{n-1}(C)) \in \{0, 1, \dots, k\}^n$ such that

$$C = B_{i_0(C)}(\omega) \cap (T_\omega^1)^{-1} B_{i_1(C)}(\vartheta\omega) \cap (T_\omega^2)^{-1} B_{i_2(C)}(\vartheta^2\omega) \cap \dots \cap (T_\omega^{n-1})^{-1} B_{i_{n-1}(C)}(\vartheta^{n-1}\omega).$$

Now fix a $C \in \mathcal{B}_n(\omega)$ and let \mathcal{Y} denote the collection of all $\tilde{C} \in \mathcal{B}_n(\omega)$ with

$$d_{\delta, n}^\omega(x(C), x(\tilde{C})) \leq 1.$$

Then we have

$$\#\{i_l(\tilde{C}) : \tilde{C} \in \mathcal{Y}\} \leq 2, \quad l = 0, 1, \dots, n-1. \quad (4.5)$$

To see this inequality, we assume on the contrary that there are three elements $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \in \mathcal{Y}$ corresponding to the distinct values $i_l(\tilde{C}_1), i_l(\tilde{C}_2), i_l(\tilde{C}_3)$ for some $0 \leq l \leq n-1$ respectively. Then without loss of generality, we may assume $i_l(\tilde{C}_1) \neq 0$ and $i_l(\tilde{C}_2) \neq 0$. This implies

$$\begin{aligned} d_{\delta, n}^\omega(x(\tilde{C}_1), x(\tilde{C}_2)) &\geq d(T_\omega^l x(\tilde{C}_1), T_\omega^l x(\tilde{C}_2)) \delta^{-1} \geq d(B_{i_l(\tilde{C}_1)}(\vartheta^l\omega), B_{i_l(\tilde{C}_2)}(\vartheta^l\omega)) \delta^{-1} \\ &\geq d(B_{i_l(\tilde{C}_1)}, B_{i_l(\tilde{C}_2)}) \delta^{-1} \geq b\delta^{-1} > 2 \\ &\geq d_{\delta, n}^\omega(x(\tilde{C}_1), x(\tilde{C})) + d_{\delta, n}^\omega(x(\tilde{C}), x(\tilde{C}_2)). \end{aligned}$$

Which leads to a contradiction, thus (4.5) is true, from which the claim follows. The third inequality follows from the fact that $B_{i_l(\tilde{C}_j)}(\vartheta^l\omega) = B_{i_l(\tilde{C}_j)} \cap \mathcal{E}_{\vartheta^l\omega} \subseteq B_{i_l(\tilde{C}_j)} (j = 1, 2)$.

In the following we will construct an (ω, δ, n) -separated set G of \mathcal{E}_ω for T such that

$$2^n \sum_{y \in G} e^{f_n(\omega, y)} \geq \sum_{C \in \mathcal{B}_n(\omega)} e^{f_n(\omega, x(C))}. \quad (4.6)$$

(I) Take an element $C_1 \in \mathcal{B}_n(\omega)$ such that $f_n(\omega, x(C_1)) = \max_{C \in \mathcal{B}_n(\omega)} f_n(\omega, x(C))$. Let \mathcal{Y}_1 denote the collection of all $\tilde{C} \in \mathcal{B}_n(\omega)$ with $d_{\delta, n}^\omega(x(\tilde{C}), x(C_1)) \leq 1$. Then the cardinality of \mathcal{Y}_1 does not exceed 2^n .

(II) If the collection $\mathcal{B}_n(\omega) \setminus \mathcal{Y}_1$ is not empty, we choose an element $C_2 \in \mathcal{B}_n(\omega) \setminus \mathcal{Y}_1$ such that $f_n(\omega, x(C_2)) = \max_{C \in \mathcal{B}_n(\omega) \setminus \mathcal{Y}_1} f_n(\omega, x(C))$. Let \mathcal{Y}_2 denote the collection of $\tilde{C} \in \mathcal{B}_n(\omega) \setminus \mathcal{Y}_1$ with $d_{\delta, n}^\omega(x(\tilde{C}), x(C_2)) \leq 1$. We continue this process. More precisely in step m , we choose an element $C_m \in \mathcal{B}_n(\omega) \setminus \bigcup_{j=1}^{m-1} \mathcal{Y}_j$ such that

$$f_n(\omega, x(C_m)) = \max_{C \in \mathcal{B}_n(\omega) \setminus \bigcup_{j=1}^{m-1} \mathcal{Y}_j} f_n(\omega, x(C)).$$

Let \mathcal{Y}_m denote the set of all $\tilde{C} \in \mathcal{B}_n(\omega) \setminus \bigcup_{j=1}^{m-1} \mathcal{Y}_j$ with $d_{\delta,n}^\omega(x(\tilde{C}), x(C_m)) \leq 1$. Since the partition $\mathcal{B}_n(\omega)$ is finite, the above process will stop at some step l . Denote $G = \{x(C_j) : 1 \leq j \leq l\}$. Then G is a (ω, δ, n) -separated set and

$$\sum_{y \in G} e^{f_n(\omega, y)} = \sum_{j=1}^l e^{f_n(\omega, x(C_j))} \geq \sum_{j=1}^l 2^{-n} \sum_{C \in \mathcal{Y}_j} e^{f_n(\omega, x(C))} = 2^{-n} \sum_{C \in \mathcal{B}_n(\omega)} e^{f_n(\omega, x(C))},$$

from which (4.6) follows.

Let $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$. Then

$$\begin{aligned} & H_{\mu_\omega}(\mathcal{B}_n(\omega)) + \int_{\mathcal{E}_\omega} f_n(\omega, x) d\mu_\omega(x) \\ & \leq \sum_{C \in \mathcal{B}_n(\omega)} \mu_\omega(C) (f_n(\omega, x(C)) - \log \mu_\omega(C)) \\ & \leq \log \sum_{C \in \mathcal{B}_n(\omega)} e^{f_n(\omega, x(C))} \\ & \leq \log 2^n \sum_{y \in G} e^{f_n(\omega, y)} \\ & = n \log 2 + \log \sum_{y \in G} e^{f_n(\omega, y)}, \end{aligned}$$

the second inequality follows from the standard inequality: $\sum p_i(a_i - \log p_i) \leq \log \sum e^{a_i}$ for any probability vector (p_1, p_2, \dots, p_m) , and the equality holds if and only if $p_i = e^{a_i} / \sum e^{a_j}$. Integrating against \mathbb{P} on both sides of the above inequality, and dividing by n , we have

$$\frac{1}{n} \int H_{\mu_\omega}(\mathcal{B}_n(\omega)) d\mathbb{P}(\omega) + \frac{1}{n} \int f_n(\omega, x) d\mu(\omega, x) \leq \log 2 + \frac{1}{n} \int \log \sum_{y \in G} e^{f_n(\omega, y)} d\mathbb{P}(\omega).$$

Letting $n \rightarrow \infty$, we obtain

$$h_\mu^{(r)}(T, \Omega \times \mathcal{B}) + \mathcal{F}_*(\mu) \leq \log 2 + \pi_T(\mathcal{F})(\delta).$$

Using corollary 3.2 in [3], we have

$$\begin{aligned} h_\mu^{(r)}(T, \Omega \times \mathcal{A}) + \mathcal{F}_*(\mu) & \leq h_\mu^{(r)}(T, \Omega \times \mathcal{B}) + \int H_{\mu_\omega}(\mathcal{A}(\omega) \mid \mathcal{B}(\omega)) d\mathbb{P} + \mathcal{F}_*(\mu) \\ & \leq \log 2 + \alpha + \pi_T(\mathcal{F})(\delta). \end{aligned}$$

Since this is true for all \mathcal{A} , α and δ , we know

$$h_\mu^{(r)}(T) + \mathcal{F}_*(\mu) \leq \log 2 + \pi_T(\mathcal{F}).$$

Applying the above argument to T^n and $\mathcal{F}^{(n)}$, since $h_\mu^{(r)}(T^n) = nh_\mu^{(r)}(T)$ (see theorem 3.6 in [3]), and using lemma 3.3, we obtain

$$\begin{aligned} n(h_\mu^{(r)}(T) + \mathcal{F}_*(\mu)) &\leq \log 2 + \pi_{T^n}(\mathcal{F}^{(n)}) \\ &\leq \log 2 + n\pi_T(\mathcal{F}). \end{aligned}$$

Since n is arbitrary, we have $h_\mu^{(r)}(T) + \mathcal{F}_*(\mu) \leq \pi_T(\mathcal{F})$.

Step 2: If $\pi_T(\mathcal{F}) \neq -\infty$, then for any small enough $\epsilon > 0$, there exists a $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$ such that $\mathcal{F}_*(\mu) \neq -\infty$ and $h_\mu^{(r)}(T) + \mathcal{F}_*(\mu) \geq \pi_T(\mathcal{F})(\epsilon)$.

Let $\epsilon > 0$ be an arbitrary small number such that $\pi_T(\mathcal{F})(\epsilon) \neq -\infty$. For any $n \in \mathbb{N}$, due to lemma 3.1, we can take a measurable in ω family of maximal (ω, ϵ, n) separated sets $G(\omega, \epsilon, n) \subset \mathcal{E}_\omega$ such that

$$\sum_{x \in G(\omega, \epsilon, n)} e^{f_n(\omega, x)} \geq \frac{1}{e} \pi_T(\mathcal{F})(\omega, \epsilon, n). \quad (4.7)$$

Next, define probability measures $v^{(n)}$ on \mathcal{E} via their measurable disintegrations

$$v_\omega^{(n)} = \frac{\sum_{x \in G(\omega, \epsilon, n)} e^{f_n(\omega, x)} \delta_x}{\sum_{y \in G(\omega, \epsilon, n)} e^{f_n(\omega, y)}}$$

where δ_x denotes the Dirac measure at x , so that $dv^{(n)}(\omega, x) = dv_\omega^{(n)}(x) d\mathbb{P}(\omega)$, and set

$$\mu^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} \Theta^i v^{(n)}.$$

By the definition of $\pi_T(\mathcal{F})(\epsilon)$ and lemma 3.2 (i)-(ii), we can choose a subsequence of positive integers $\{n_j\}$ such that

$$\lim_{j \rightarrow \infty} \frac{1}{n_j} \int \log \pi_T(\mathcal{F})(\omega, \epsilon, n_j) d\mathbb{P}(\omega) = \pi_T(\mathcal{F})(\epsilon) \quad \text{and } \mu^{(n_j)} \Rightarrow \mu \text{ as } j \rightarrow \infty \quad (4.8)$$

for some $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$.

Now we choose a partition $\mathcal{A} = \{A_1, \dots, A_k\}$ of X with $\text{diam}(\mathcal{A}) := \max\{\text{diam}(A_j) : 1 \leq j \leq k\} \leq \epsilon$ and such that $\int \mu_\omega(\partial A_i) d\mathbb{P}(\omega) = 0$ for all $1 \leq i \leq k$, where ∂ denotes the boundary. Set $\mathcal{A}(\omega) = \{A_1(\omega), \dots, A_k(\omega)\}$, $A_i(\omega) = A_i \cap \mathcal{E}_\omega$, $1 \leq i \leq k$. Since each element of $\bigvee_{i=0}^{n-1} (T_\omega^i)^{-1} \mathcal{A}(\vartheta^i \omega)$ contains at most one element of $G(\omega, \epsilon, n)$, we have

by (4.7),

$$\begin{aligned}
& H_{v_\omega^{(n)}}\left(\bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\mathcal{A}(\vartheta^i\omega)\right) + \int f_n(\omega, x)dv_\omega^{(n)}(x) \\
&= \sum_{y \in G(\omega, \epsilon, n)} v_\omega^{(n)}(\{y\})(f_n(\omega, y) - \log v_\omega^{(n)}(\{y\})) \\
&= \log \sum_{y \in G(\omega, \epsilon, n)} e^{f_n(\omega, y)} \\
&\geq \log \pi_T(\mathcal{F})(\omega, \epsilon, n) - 1.
\end{aligned}$$

Let $\mathcal{B} = \{B_1, \dots, B_k\}$, $B_i = (\Omega \times A_i) \cap \mathcal{E}$. Then \mathcal{B} is a partition of \mathcal{E} and $B_i(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in B_i\} = A_i(\omega)$. Integrating in the above inequality against \mathbb{P} and dividing by n , we have by (2.2) the inequality

$$\frac{1}{n}H_{v^{(n)}}\left(\bigvee_{i=0}^{n-1}(\Theta^i)^{-1}\mathcal{B} \mid \mathcal{W}_\mathcal{E}\right) + \frac{1}{n}\int f_n dv^{(n)} \geq \frac{1}{n}\int \log \pi_T(\mathcal{F})(\omega, \epsilon, n) d\mathbb{P}(\omega) - \frac{1}{n}. \quad (4.9)$$

Consider $q, n \in \mathbf{N}$ such that $1 < q < n$ and for $0 \leq l < q$ and let $a(l)$ denote the integer part of $(n - l)q^{-1}$, so that $n = l + a(l)q + r$ with $0 \leq r < q$. Then

$$\bigvee_{i=0}^{n-1}(\Theta^i)^{-1}\mathcal{B} = \left(\bigvee_{j=0}^{a(l)-1}(\Theta^{l+jq})^{-1}\bigvee_{i=0}^{q-1}(\Theta^i)^{-1}\mathcal{B}\right) \vee \bigvee_{m \in S_l}(\Theta^m)^{-1}\mathcal{B},$$

where S_l is a subset of $\{0, 1, \dots, n - 1\}$ with cardinality at most $2q$. Since $\text{card}\mathcal{B} = k$, taking into account the subadditivity of conditional entropy (see [10], section 2.1) it follows that

$$H_{v^{(n)}}\left(\bigvee_{i=0}^{n-1}(\Theta^i)^{-1}\mathcal{B} \mid \mathcal{W}_\mathcal{E}\right) \leq \sum_{j=0}^{a(l)-1} H_{\Theta^{l+jq}v^{(n)}}\left(\bigvee_{i=0}^{q-1}(\Theta^i)^{-1}\mathcal{B} \mid \mathcal{W}_\mathcal{E}\right) + 2q \log k.$$

Summing here over $l \in \{0, 1, \dots, q - 1\}$, we have

$$\begin{aligned}
qH_{v^{(n)}}\left(\bigvee_{i=0}^{n-1}(\Theta^i)^{-1}\mathcal{B} \mid \mathcal{W}_\mathcal{E}\right) &\leq \sum_{m=0}^{n-1} H_{\Theta^m v^{(n)}}\left(\bigvee_{i=0}^{q-1}(\Theta^i)^{-1}\mathcal{B} \mid \mathcal{W}_\mathcal{E}\right) + 2q^2 \log k \\
&\leq nH_{\mu^{(n)}}\left(\bigvee_{i=0}^{q-1}(\Theta^i)^{-1}\mathcal{B} \mid \mathcal{W}_\mathcal{E}\right) + 2q^2 \log k
\end{aligned}$$

where the second inequality relies on the general property of the conditional entropy of partitions $H_{\sum_i p_i \eta_i}(\xi \mid \mathcal{R}) \geq \sum_i p_i H_{\eta_i}(\xi \mid \mathcal{R})$ which holds for any finite partition ξ , σ -algebra \mathcal{R} , probability measures η_i , and probability vector (p_i) , $i = 1, \dots, n$, in view

of the convexity of $t \log t$ in the same way as in the unconditional case (see [17], pp. 183 and 188). Dividing by nq in inequality as above, we have

$$\frac{1}{n} H_{v^{(n)}} \left(\bigvee_{i=0}^{n-1} (\Theta^i)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right) \leq \frac{1}{q} H_{\mu^{(n)}} \left(\bigvee_{i=0}^{q-1} (\Theta^i)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right) + \frac{2q \log k}{n}.$$

In particular, we have

$$\frac{1}{n_i} H_{v^{(n_i)}} \left(\bigvee_{j=0}^{n_i-1} (\Theta^j)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right) \leq \frac{1}{q} H_{\mu^{(n_i)}} \left(\bigvee_{j=0}^{q-1} (\Theta^j)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right) + \frac{2q \log k}{n_i}. \quad (4.10)$$

Observe that the boundary of $\bigvee_{i=0}^{q-1} (T_{\omega}^i)^{-1} \mathcal{A}(\vartheta^i \omega)$ is contained in the union of boundaries of $(T_{\omega}^i)^{-1} \mathcal{A}(\vartheta^i \omega)$ and $\mu_{\omega}((T_{\omega}^i)^{-1} \partial \mathcal{A}(\vartheta^i \omega)) = \mu_{\vartheta^i \omega}(\partial \mathcal{A}(\vartheta^i \omega)) \mathbb{P} - a.s.$. $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$ implies that $\mu_{\omega}(\partial \bigvee_{i=0}^{q-1} (T_{\omega}^i)^{-1} \mathcal{A}(\vartheta^i \omega)) = 0 \quad \mathbb{P} - a.s.$. Taking into account lemma 3.2(iii), we have

$$\limsup_{i \rightarrow \infty} \frac{1}{q} H_{\mu^{(n_i)}} \left(\bigvee_{j=0}^{q-1} (\Theta^j)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right) \leq \frac{1}{q} H_{\mu} \left(\bigvee_{j=0}^{q-1} (\Theta^j)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right).$$

Letting i approach ∞ in (4.10), we have

$$\limsup_{i \rightarrow \infty} \frac{1}{n_i} H_{v^{(n_i)}} \left(\bigvee_{j=0}^{n_i-1} (\Theta^j)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right) \leq \frac{1}{q} H_{\mu} \left(\bigvee_{j=0}^{q-1} (\Theta^j)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right). \quad (4.11)$$

From lemma 3.5, we know

$$\limsup_{i \rightarrow \infty} \frac{1}{n_i} \int f_{n_i} d\mu^{(n_i)} \leq \mathcal{F}_*(\mu). \quad (4.12)$$

Combining (4.8), (4.9), (4.11) with (4.12), we obtain

$$\frac{1}{q} H_{\mu} \left(\bigvee_{j=0}^{q-1} (\Theta^j)^{-1} \mathcal{B} \mid \mathcal{W}_{\mathcal{E}} \right) + \mathcal{F}_*(\mu) \geq \pi_T(\mathcal{F})(\epsilon).$$

Letting $q \rightarrow \infty$, we have

$$\pi_T(\mathcal{F})(\epsilon) \leq h_{\mu}^{(r)}(T, \mathcal{B}) + \mathcal{F}_*(\mu) \leq h_{\mu}^{(r)}(T) + \mathcal{F}_*(\mu).$$

This completes the proof of step 2.

Step 3: $\pi_T(\mathcal{F}) = -\infty$ if and only if $\mathcal{F}_*(\mu) = -\infty$ for all $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$.

By step 1 we have $\pi_T(\mathcal{F}) \geq h_{\mu}^{(r)}(T) + \mathcal{F}_*(\mu)$ for all $\mu \in \mathcal{M}_{\mathbb{P}}^1(\mathcal{E}, T)$ with $\mathcal{F}_*(\mu) \neq -\infty$, which shows the necessity. The sufficiency is implied by step 2 (since if $\pi_T(\mathcal{F}) \neq -\infty$, then by step 2 there exists some μ with $\mathcal{F}_*(\mu) \neq -\infty$). This completes the proof of the theorem. \square

5 The Hausdorff dimension for asymptotic conformal repellers

In this section, we consider the Hausdorff dimension for repeller in random dynamical system(RDS). Precisely, fix an ergodic invertible transformation ϑ of a probability space $(\Omega, \mathcal{W}, \mathbb{P})$ and let M be a compact Riemann manifold. We consider a measurable family $T = \{T_\omega : M \rightarrow M\}$ of C^1 maps, i.e. $(\omega, x) \mapsto T_\omega x$ is assumed to be measurable. This determines a differentiable RDS via $T_\omega^n = T_{\vartheta^{n-1}\omega} \cdots T_{\vartheta\omega} T_\omega$, $n \geq 1$. $\mathcal{E} \subset \Omega \times M$ is a measurable set and such that all the fibers (sometimes called ω -sections) $\mathcal{E}_\omega = \{x \in X \mid (\omega, x) \in \mathcal{E}\}$ are compact. \mathcal{E} is said to be invariant with T if $T_\omega \mathcal{E}_\omega = \mathcal{E}_{\vartheta\omega}$ $\mathbb{P}-a.s$. In [12], [4], the authors consider the Hausdorff dimension for repeller in $C^{1+\alpha}$ conformal random dynamical system. They prove that, if T_ω is $C^{1+\alpha}$ conformal for $\mathbb{P}-a.s$ and $\mathcal{E} \subset \Omega \times M$ is a repeller which is invariant with T for random dynamical system, then the Hausdorff dimension can be obtained as the zero t_0 of $t \mapsto \pi_T(-t \log \|D_x T\|)$, where $\pi_T(-t \log \|D_x T\|)$ is topological pressure for random dynamical system T with additive potential $-t \log \|D_x T\|$.

A repeller is called conformal if T_ω for $\mathbb{P}-a.s$ is conformal. In some sense, conformality in random dynamical systems is strong. Now we give a definition of asymptotically conformal repeller, which is weaker than conformal repeller. Let $\mathcal{M}_\mathbb{P}(\mathcal{E})$ be the set of T invariant probability measures on \mathcal{E} whose marginal on Ω coincides with \mathbb{P} and $E_\mathbb{P}(\mathcal{E})$ be the set of T invariant ergodic probability measures on \mathcal{E} whose marginal on Ω coincides with \mathbb{P} . By the Oseledec multiplicative ergodic theorem [14], for any $\mu \in E_\mathbb{P}(\mathcal{E})$, we can define Lyapunov exponents $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_d(\mu)$, $d = \dim M$. An invariant repeller for random dynamical system is called asymptotically conformal if for any $\mu \in E_\mathbb{P}(\mathcal{E})$, $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_d(\mu)$. It is obvious that a conformal repeller is an asymptotically conformal repeller, but reverse isn't true. Using topological pressure of random bundle transformations in sub-additive case, we can obtain the Hausdorff dimension for asymptotically conformal repeller. We state the result as follows, and the proof will be given in the forthcoming paper.

Theorem 5.1. *Let T be $C^{1+\alpha}$ random dynamical system and \mathcal{E} be an asymptotically conformal repeller. Then the Hausdorff dimension of \mathcal{E} is zero t^* of $t \mapsto \pi_T(-t\mathcal{F})$, where $\mathcal{F} = \{\log m(D_x T_\omega^n), (\omega, x) \in \mathcal{E}, n \in \mathbb{N}\}$ and $m(A) = \|A^{-1}\|^{-1}$.*

Remark 2. *If \mathcal{E} isn't asymptotically conformal repeller, we can obtain the upper estimate of the Hausdorff dimension by using topological pressure of random bundle transformations in sub-additive case.*

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